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HEAT CONDUCTION IN SOLIDS WITH RANDOM EXTERNAL TEMPERATURES AND/OR RANDOM INTERNAL HEAT GENERATION*

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Abstract—Several problems are studied in the theory of heat conduction in solids in which the internally generated heat and/or the surrounding temperature are stochastic functions of position and time. For linear systems subjected to random disturbances which form a Gaussian process, it is sufficient to determine the mean and mean square value of the temperature field to fix its complete statistics. This procedure is carried through for the several problems considered. It is found that the convergence problems for the infinite series for the mean square temperature (under white noise disturbance) are much less severe than those encountered in random vibration theory where the governing partial differential equations are of hyperbolic type rather than parabolic as in heat conduction. Numerical calculations are carried out with a digital computer in some cases.

Some comments on the problem of the earth's temperature are made.

NOMENCLATURE

κ, diffusivity;
θ, $= \dfrac{Q}{\rho c}$ heat generation per unit volume divided by ρc;
ρ, material density;
c, specific heat;
$v(x^i, t)$, temperature distribution;
∇^2, Laplacian operator;
x^i, space coordinates (x, y, z) $i = 1, 2, 3$;
t, time;
E, expectation operator (mathematical expectation);
$\phi(x^i, t)$, random scalar field;
R_ϕ, autocorrelation function of $\phi(x^i, t)$;
ξ^i, space point;
τ, time instant, also κt;
∇_1^3, Laplacian operator at the point x_1^i;
∇_2^2, Laplacian operator at the point x_2^i;
Q, heat generated per unit volume;
l, slab width or wire length;
S, region occupied by the slab or wire;
q, $Q/\rho c\kappa$;
W, weighting function;
W_n, weighting function of the system for the nth mode;
T, time limit;
S_o, constant spectral density;
h, time interval, heat-transfer coefficients;
β, constant in markoff correlation;
$\langle Q_o^2\rangle$, mean square heat generated;
$\lambda(\eta)$, a particular function;
$u(x^i, t)$, difference between solid's temperature and the surrounding;
$T(\tau)$, temperature of surrounding;
d, time interval;
$\langle T_o^2\rangle$, mean square temperature of surroundings;
r, radial coordinate;
a, radius of sphere;
$F(\tau)$, given temperature function;
$S_n(\omega)$, power spectral density of $x(t)$;
ω, angular frequency;
ω_1, frequency parameter.

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1. INTRODUCTION

In recent years, there have been increasing applications of probabilistic methods to engineering problems involving random functions [1–10]. For linear systems with Gaussian processes as forcing functions, a complete solution (all of the multivariate probability distribution functions of the random field or output) is available [11–13]. The central aim in the analysis of Gaussian random processes is to obtain the autocorrelation function of the output, for this quantity determines the entire output process (assuming zero mean).

Considerable work with the Gaussian random process has been done with Brownian motion in physics [1] and with noise in electrical engineering [2]. It appears, however, that only recently has much attention been given to the application of the Gaussian process to random problems of an essentially mechanical engineering character.

In this paper, several problems in heat conduction involving random functions are solved. Our problem will be the determination of the autocorrelation function of the temperature distribution in terms of the autocorrelation function of the exciting function. Since we are treating only stationary Gaussian processes for the exciting functions, the autocorrelation function of the temperature (which gives us the mean square temperature) will be sufficient (assuming zero mean) to determine the first probability function of the temperature field.

2. GENERAL THEORY

Before going to specific problems, it is important to make some general remarks about random problems of heat conduction. The basic equation for the linear conduction of heat in homogeneous isotropic solids is

$$\kappa \nabla^2 v(x^i, t) + \theta(x^i, t) = \frac{\partial v(x^i, t)}{\partial t} \qquad (1)^*$$

Heat conduction problems are in general boundary value and initial value problems combined. The temperature field may become a random field in any of five ways:

(1) random heat generation,
(2) random boundary values,
(3) random initial conditions,
(4) random material properties, and
(5) random geometry.

In the subsequent study, problems in the first and second categories are considered.

When we consider random functions, we will always understand that there is an underlying probability space on which a probability measure is defined. This will enable us to carry our problem through entirely in a probability-theoretical manner making unnecessary any appeal to the ergodic hypothesis. We will, however, need to modify our random functions in order that certain integral transforms will exist. The manner in which this can be done is through the use of truncated functions [14].

The autocorrelation function of a random field $\phi(x^i, t)$ with zero mean value is defined as

$$R_\phi(x^i, \xi^i, t, \tau) = E\{\phi(x^i, t)\, \phi(\xi^i, \tau)\} \quad i = 1, 2, 3$$

in which E is the expectation operator [15]. The value of the autocorrelation function at $\xi^i = x^i$ and $\tau = t$ is the mean square value of the random field. The importance of the autocorrelation function lies in the fact that, for linear systems, the autocorrelation functions of the output and input are simply related. It is also easy to show that the autocorrelation function of the temperature field will satisfy the following partial differential equation

$$\kappa^2 \nabla_2^2 \nabla_1^2 R_v - \kappa \nabla_1^2 \frac{\partial R_v}{\partial t_2} - \kappa \nabla_2^2 \frac{\partial R_v}{\partial t_1} + \frac{\partial^2 R_v}{\partial t_1 \partial t_2} = R_\theta \qquad (2)$$

where

$$R_v = E\{v(x_1^i, t_1)\, v(x_2^i, t_2)\}$$
$$R_\theta = E\{\theta(x_1^i, t_1)\, \theta(x_2^i, t_2)\} \qquad (3)$$

Equations for higher order correlations could also be obtained but usually it is simpler to solve (1) under appropriate boundary and initial conditions and then form the various correlation functions from the solution. The latter procedure can be justified with the use of generalized harmonic analysis [14].

* The term $\partial v(x^i, t)/\partial t$ plays the same role as velocity damping plays in random vibration theory [6].

In the problems involving normal random processes, the determination of R_v will be sufficient. We now apply the theory to some problems in heat conduction which may be of some practical interest.

3. ONE DIMENSIONAL PROBLEMS

(*a*) *Random heat conduction in a thin wire or a solid bounded by two parallel planes*

Consider a thin wire of length l or an infinite slab of thickness l, in which heat is being generated in each volume element in a random fashion. Such might be the case if a noise electric current flows through the wire or slab.

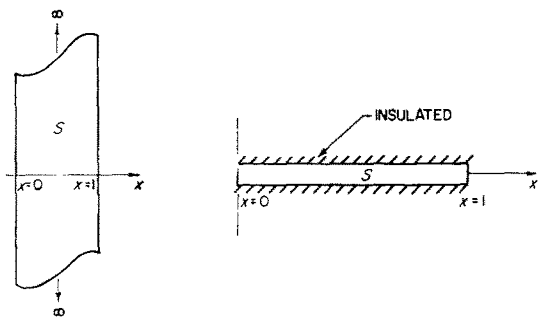

FIG. 1. Random heat conduction in a slab or an insulated wire.

The temperature distribution is given by the classical heat equation (for a wire of infinitesimal diameter)

$$\kappa \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} - \frac{Q}{\rho c} \tag{4}$$

$Q(x, t)$ is the heat generated per unit volume and is a random function. $v(x, t)$ is the temperature, κ is the diffusivity, ρ is the material density and c is the specific heat capacity of the material. We shall assume that $Q(x, t)$ is a stationary Gaussian random process, then, because (4) is a linear equation, the temperature distribution $v(x, t)$ will also be a Gaussian process [13]. We seek the RMS value of $v(x, t)$. From the general theory, we know that the autocorrelation function of $Q(x, t)$ is sufficient to calculate the RMS value of $v(x, t)$. We now consider the problem of determining the temperature distribution with random generation of heat in the medium, the ends or faces being maintained at zero temperature. The boundary value problem is

$$\kappa \frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t} - \frac{Q}{\rho c} \tag{5}$$

$$\text{in } S \quad v(0, t) = v(l, t) = 0$$

To solve, we let

$$\kappa t = \tau$$

and

$$Q/\rho c \kappa = q(x, \tau),$$

then

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial \tau} - q \tag{6}$$

Take the finite sine transform of (6) to get

$$\frac{\mathrm{d}v_n}{\mathrm{d}\tau} + \left(\frac{n\pi}{l}\right)^2 v_n = q_n(\tau) \tag{7}$$

where

$$v_n(\tau) = \int_0^l v(x^1, \tau) \sin \frac{n\pi x^1}{l} \, \mathrm{d}x^1$$

and

$$q_n(\tau) = \int_0^l q(x^1, \tau) \sin \frac{n\pi x^1}{l} \, \mathrm{d}x^1$$

The Fourier transform of (7) with respect to τ is

$$- i\zeta \bar{v}_n(\zeta) + \left(\frac{n\pi}{l}\right)^2 \bar{v}_n(\zeta) = \bar{q}_n(\zeta)$$

where truncation [14] is used on $v_n(\tau)$ and $q_n(\tau)$ to insure the existence of

$$\bar{v}_n(\zeta) = \int_{-\infty}^{\infty} \exp\left[- i\zeta\tau\right] v_n(\tau) \, \mathrm{d}\tau$$

and

$$\bar{q}_n(\zeta) = \int_{-\infty}^{\infty} \exp\left[- i\zeta\tau\right] q_n(\tau) \, \mathrm{d}\tau$$

From transform of (7), we find

$$\bar{v}_n(\zeta) = \frac{\bar{q}_n(\zeta)}{i\zeta + (n\pi/l)^2} \tag{8}$$

By applying the inversion theorem for Fourier transforms to (8), we get

$$v(x, \tau) = \frac{2}{l} \sum_{n=1}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\,[i\tau\zeta]\, \frac{\bar{q}_n(\zeta)}{i\zeta + (n\pi/l)^2}\, \mathrm{d}\zeta \,.\, \sin \frac{n\pi x}{l}$$

or

$$v(x, \tau) = \frac{2}{l} \sum_{n=1}^{\infty} \int_{0}^{l} \sin \frac{n\pi x^1}{l}\, \mathrm{d}x^1 \,. \int_{-\infty}^{\infty} q(x^1, \tau - \lambda)\, W_n(\lambda)\, \mathrm{d}\lambda \,.\, \sin \frac{n\pi x}{l} \tag{9}$$

where

$$W_n(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp\,[i\lambda\zeta]}{i\zeta + (n\pi/l)^2}\, \mathrm{d}\zeta = \begin{cases} \exp\,[-\,(n\pi/l)^2\, \lambda], & \lambda > 0 \\ 0, & \lambda < 0 \end{cases}$$

is the usual weighting function for the nth mode of the system.

The autocorrelation function of the output is given by

$$\left. \begin{aligned} R_v = \frac{4}{l^2} \sum_{n,m=1}^{\infty} \Bigg\{ \int_0^l \int_0^l \sin \frac{n\pi x^1}{l} \sin \frac{m\pi \xi^1}{l}\, \mathrm{d}x^1 \mathrm{d}\xi^1 \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_q(x^1, \xi^1; h + s - r) \,. \\ W_n(s)\, W_m(r)\, \mathrm{d}s\, \mathrm{d}r \Bigg\} \sin \frac{n\pi x}{l} \sin \frac{m\pi \xi}{l} \end{aligned} \right\} \tag{10}$$

where

$$R_q(x, \xi, h) = E\{q(x, \tau)\, q(\xi, \tau + h)\}$$

is the autocorrelation function of $q(x, \tau)$. We also used the fact that $q(x, \tau)$ is a stationary random function of time. R_q, for stationary processes under suitable conditions, is also given by the temporal average

$$\lim_{T \to \infty} \frac{1}{2T} \int_{T}^{T} q(x, \tau)\, q(\xi, \tau + h)\, \mathrm{d}\tau$$

with probability one.

In general, $R_q(x, \xi, h)$ must be determined from the given experimental temporal record of $q(x, \tau)$. However, some special simple cases are of theoretical interest:

(*i*) $$R_q = \frac{S_o}{\kappa^2 \rho^2 c^2}\, \delta(x - \xi)\, \delta(h),$$

purely random process. S_o is a constant. $\langle q \rangle = 0$. Then

$$R_v = \frac{S_o l}{\pi^2 \kappa^2 \rho^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \exp \left[-\left(\frac{n\pi}{l}\right)^2 | \quad | \right] \times \sin \frac{n\pi x}{l} \sin \frac{n\pi \xi}{l} \tag{11}$$

The mean square temperature is obtained by setting $x = \xi$ and $h = 0$ in (11). We then have

$$\langle v^2 \rangle = \frac{S_o l}{\pi^2 \kappa^2 \rho^2 c^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin^2 \frac{n\pi x}{l}, \quad \langle v \rangle = 0$$

or

$$\langle v^2 \rangle = \frac{S_o l}{2\kappa^2 \rho^2 c^2} \left(\frac{x}{l}\right) \left(1 - \frac{x}{l}\right), \quad \langle v \rangle = 0. \tag{12}$$

This result is plotted in Fig. 2.

Of course the probability density function is found from

$$p(v) = \frac{1}{\sqrt{(2\pi)}\, v_{\mathrm{RMS}}} \exp \left[-\frac{v^2}{2\langle v^2 \rangle} \right] \tag{13}$$

where $v_{\mathrm{RMS}} = \sqrt{\langle v^2 \rangle}$ is the root mean square temperature.

(*ii*) $R_q = S_o/\kappa^2\rho^2c^2\, \delta(h)$, $\langle q \rangle = 0$, purely random process, no space dependence.

The mean square temperature is

$$\langle v^2(x) \rangle = \frac{16 S_o l^2}{\pi^4 \kappa^2 \rho^2 c^2} \sum_{n,m=1}^{\infty} \frac{1}{nm} \left(\frac{1}{n^2} + \frac{1}{m^2} \right) \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} \tag{14}$$

(*iii*) $$R_q = \frac{\langle Q_o^2 \rangle}{\kappa^2 \rho^2 c^2} \exp\,[-\,\beta\, |\, h\, |],\ \langle q \rangle = 0,$$

Markoff correlation.

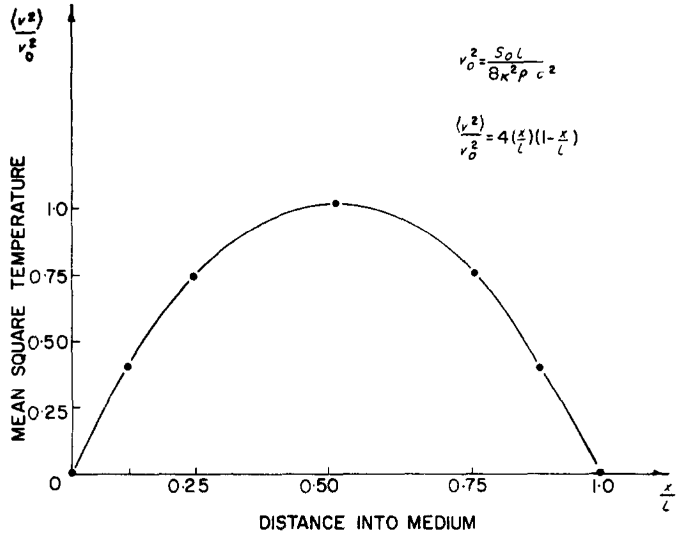

FIG. 2. Mean square temperature in the solid.

The mean square temperature is

$$\left.\begin{aligned}\langle v^2(x)\rangle = \frac{16\langle Q_o{}^2\rangle}{l^2\kappa^2\rho^2c^2}\sum_{n,m=1}^{\infty}\frac{1}{(n\pi/l)^2\,(m\pi/l^2)}\\ \left\{\frac{1}{[\beta-(m\pi/l)^2]\,[(n\pi/l)^2+(m\pi/l)^2]}\right.\\ +\frac{1}{[\beta+(m\pi/l)^2]\,[(n\pi/l)^2+(m\pi/l)^2]}\\ \left.-\frac{1}{[\beta+(n\pi/l^2)][\beta-(m\pi/l)^2]}\right\}\\ \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}\end{aligned}\right\}\quad(15)$$

$\langle Q_o{}^2\rangle$ and β are constants.

(*b*) *Radiation at the face* $x = 0$ *into a medium of randomly varying temperature; the other face extends to infinity. No heat generation. (Furnace problem.)*

The boundary value problem is

$$\left.\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{\partial v}{\partial \tau}\\ \frac{\partial v(0,\tau)}{\partial x} - h[v(0,\tau) - T(\tau)] &= 0\end{aligned}\right\}\quad(16)$$

where $T(\tau)$ is the random temperature of the medium to the left of $x = 0$ and h is a heat-transfer coefficient.

Let $u(x, \tau) = v(x, \tau) - T(\tau)$, then (16) becomes

$$\left.\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial u}{\partial \tau} + \frac{\partial T}{\partial \tau}\\ \frac{\partial u(0,\tau)}{\partial x} - hu(0,\tau) &= 0\end{aligned}\right\}\quad(16a)$$

This problem may be solved by integral transforms [16, 17]. Let

$$u_K(\eta, \tau) = \int_0^\infty u(x, \tau)\,K(\eta, x)\,dx$$

then

$$u(x, \tau) = \int_0^\infty u_K(\eta, \tau)\,K(\eta, x)\,d\eta$$

where

$$K(\eta, x) = \sqrt{\left(\frac{2}{\pi}\right)}\left[\frac{h\sin(\eta x) + \eta\cos(\eta x)}{(\eta^2 + h^2)^{\frac{1}{2}}}\right]$$

Using these results in (16a) we obtain

$$\frac{du_K}{d\tau} + \eta^2 u_K = -\frac{dT}{d\tau}\int_0^\infty K(\eta, x)\,dx \quad (17)$$

Now take the Fourier transform of (17) (using truncation if necessary) and find

$$u_K = \int_{-\infty}^{\infty} \exp[-i\zeta\tau]\, u_K(\eta, \tau)\, d\tau = \frac{-i\zeta \bar{T}(\zeta) \int_0^{\infty} K(\eta, x)\, dx}{i\zeta + \eta^2} \tag{18}$$

Applying the inversion theorems, we get

$$u(x, \tau) = -\int_0^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\tau\zeta] \times \frac{i\zeta \bar{T}(\zeta) \int_0^{\infty} K(\eta, x)\, dx}{i\zeta + \eta^2}\, d\zeta K(\eta, x)\, d\eta$$

or

$$u(x, \tau) = -\int_0^{\infty} \{ \int_{-\infty}^{\infty} T(\tau - \beta)\, W(\beta, \eta)\, d\beta \} \times \lambda(\eta)\, K(\eta, x)\, d\eta \tag{19}$$

where

$$W(\beta, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\beta\zeta] \frac{i\zeta\, d\zeta}{i\zeta + \eta^2} = \begin{cases} -\eta^2 \exp[-\eta^2\beta] + \delta(\beta^+), & \beta > 0 \\ 0, & \beta < 0 \end{cases}$$

and

$$\int_0^{\infty} K(\eta, x)\, dx = \lim_{a \to 0} \int_0^{\infty} \exp[-ax]\, K(\eta, x)\, dx = \frac{h\sqrt{(2/\pi)}}{\eta(\eta^2 + h^2)^{\frac{1}{2}}} \equiv \lambda(\eta)$$

The autocorrelation function of u is

$$R_u(x, \xi, \tau, \tau + d) = \int_0^{\infty}\int_0^{\infty} \{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_T(\tau - \beta, \tau + d - \beta') \times W(\beta, \eta)\, W(\beta', \eta')\, d\beta\, d\beta' \} \times \lambda(\eta)\, \lambda(\eta')\, K(\eta, x)\, K(\eta', \xi)\, d\eta\, d\eta' \tag{20}$$

where

$$R_T(\tau, \tau + d) = E\{T(\tau)\, T(\tau + d)\}$$

is the autocorrelation function of the external temperature. If $T(\tau)$ is a stationary process in time, then

$$\left.\begin{aligned} R_u(x, \xi, d) = \int_0^{\infty}\int_0^{\infty} \{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_T(d + \beta \\ - \beta')\, W(\beta, \eta)\, W(\beta', \eta')\, d\beta\, d\beta' \} \\ \times\, \lambda(\eta)\, \lambda(\eta')\, K(\eta, x)\, K(\eta', \xi)\, d\eta\, d\eta' \end{aligned}\right\} \tag{21}$$

and we find

$$\left.\begin{aligned} R_v(x, \xi, d) = \int_0^{\infty}\int_0^{\infty} \{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_T(d \\ + \beta' - \beta')\, \tilde{W}(\beta, \eta)\, \tilde{W}(\beta', \eta')\, d\beta\, d\beta' \} \\ \times\, \lambda(\eta)\, \lambda(\eta')\, K(\eta, x)\, K(\eta', \xi)\, d\eta\, d\eta' \end{aligned}\right\} \tag{22}$$

where

$$\tilde{W}(\beta) = \begin{cases} -\eta^2 \exp[-\eta^2\beta], & \beta > 0 \\ 0, & \beta < 0 \end{cases} \tag{23}$$

We now examine some special cases of (22).

(i) Pure random process for $T(\tau)$

$$R_T(d) = S_o \delta(d)$$

The mean square of $v(\kappa, \tau)$ is then given by

$$\langle v^2(x) \rangle = S_o \int_0^{\infty}\int_0^{\infty} \left. \frac{\exp[-\eta'^2 d]}{\eta'^2 + \eta^2} \right|_{d=0^+} \lambda(\eta)\, \lambda(\eta')\, K(\eta, x)\, K(\eta', x)\, d\eta\, d\eta'$$

or

$$\langle v^2(x) \rangle = \frac{4S_o h^2}{\pi^2} \int_0^{\infty}\int_0^{\infty} \frac{\eta\eta'}{\eta^2 + \eta'^2} \frac{[h \sin \eta x + \eta \cos \eta x]\, [h \sin \eta' x + \eta' \cos \eta' x]}{(\eta^2 + h^2)\, (\eta'^2 + h^2)}\, d\eta\, d\eta' \tag{24}$$

The mean square value of v at the surface $x = 0$ is given by

$$\langle v^2(0) \rangle = \frac{4S_o h^2}{\pi^2} \times \int_{\infty}^{0}\int_{\infty}^{0} \frac{\eta^2 \eta'^2\, d\eta\, d\eta'}{(\eta^2 + h^2)\, (\eta'^2 + h^2)\, (\eta^2 + \eta'^2)} \tag{25}$$

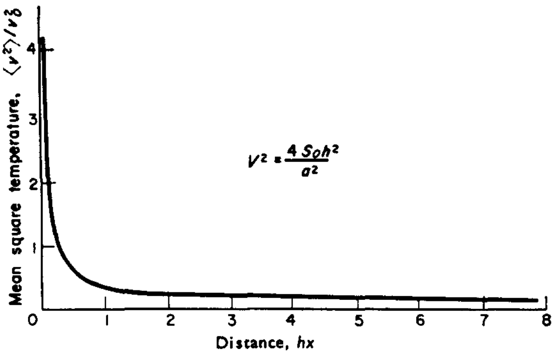

FIG. 3. Mean square temperature in a semi-infinite solid.

This expression can be evaluated by use of an electronic digital computer. A plot of equation (24) is shown in Fig. 3.

(*ii*) *Markoff correlation for* $T(\tau)$

$$R_T(d) = \langle T_o^2 \rangle \exp[-\beta |d|],$$

$\langle T_o^2 \rangle$ and β are constants.

The mean square temperature at the surface $x = 0$ is

$$\left.\begin{aligned} \langle v^2(0) \rangle &= \frac{4\langle T_o^2 \rangle h^2}{\pi^2} \int_0^\infty \int_0^\infty \\ &\left\{ \frac{\eta^2 \eta'^2}{(\eta^2 + \beta)(\eta'^2 - \beta)} - \frac{\eta^2 \eta'^2}{(\eta^2 + \eta'^2)(\eta'^2 - \beta)} \right. \\ &\left. + \frac{\eta^2 \eta'^2}{(\eta^2 + \eta'^2)(\eta'^2 + \beta)} \right\} \\ &\frac{1}{(\eta^2 + h^2)(\eta'^2 + h^2)} \,\mathrm{d}\eta\,\mathrm{d}\eta' \end{aligned}\right\} \quad (26)$$

4. PROBLEMS OF THE SPHERE (ONE DIMENSIONAL)

We consider the following problems of the sphere:

(*a*) *Radiation at the surface into a medium of randomly varying temperature*

Our problem is

$$\frac{\partial^2 \{rv(r,\tau)\}}{\partial r^2} = \frac{\partial \{rv(r,\tau)\}}{\partial \tau} \quad (27)$$

$$-\frac{\partial v}{\partial r} + h\{v - T(\tau)\} = 0, \quad \text{at } r = a$$

where $\tau = \kappa t$ and a = radius of the sphere. We also require that $v(r, \tau)$ be bounded at $r = 0$. It is convenient to transform (27) by setting $u(r, \tau) = rv(r, \tau)$. Then

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \frac{\partial u}{\partial \tau} \\ -\frac{\partial u(a,\tau)}{\partial r} + h[u(a,\tau) - aT(\tau)] &= 0 \\ u(0,\tau) &= 0 \end{aligned}\right\} \quad (28)$$

The boundary value problem in (28) can also be interpreted as finding the temperature distribution in a slab of thickness a with radiation at $x = a$ and the face $x = 0$ maintained at zero temperature.

To solve (28), we first take the Fourier transform with respect to τ to obtain

$$\left.\begin{aligned} \frac{\mathrm{d}^2 \bar{u}}{\mathrm{d}r^2} - i\zeta\bar{u} &= 0 \\ \frac{\mathrm{d}\bar{u}}{\mathrm{d}r} - h\bar{u} &= -ah\bar{T}(\zeta) \\ u(0,\zeta) &= 0 \end{aligned}\right\} \quad (29)$$

where

$$\bar{u}(r,\zeta) = \int_{-\infty}^{\infty} \exp[-i\zeta\tau]\, u(r,\tau)\,\mathrm{d}\tau$$

and

$$\bar{T}(\zeta) = \int_{-\infty}^{\infty} \exp[-i\zeta\tau]\, T(\tau)\,\mathrm{d}\tau$$

The solution to (29) is

$$\bar{u}(r,\zeta) = \frac{-ah\bar{T}(\zeta)\{\exp[\sqrt{(i\zeta)}\,r] - \exp[-\sqrt{(i\zeta)}\,r]\}}{[(-h + \sqrt{(i\zeta)}]) \exp[\sqrt{(i\zeta)}\,a] - [(-h - \sqrt{(i\zeta)}] \exp[-\sqrt{(i\zeta)}\,a]}$$

or

$$\bar{u}(r,\zeta) = \frac{-ah\bar{T}(\zeta)\sinh\sqrt{(i\zeta)}\,r}{-h\sinh\sqrt{(i\zeta)}\,a + \sqrt{(i\zeta)}\cosh\sqrt{(i\zeta)}\,a} \quad (30)$$

By the convolution theorem for Fourier transforms we obtain

$$u(r, \tau) = - ah \int_{-\infty}^{\infty} T(\tau - \beta)\, W(\beta, r)\, d\beta \qquad (31)$$

$$W(\beta, r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\beta\zeta] \times \frac{\sinh \sqrt{(i\zeta)}\, r}{- h \sinh \sqrt{(i\zeta)}\, a + \sqrt{(i\zeta)} \cosh \sqrt{(i\zeta)}\, a}\, d\zeta \qquad (32)$$

The evaluation of $W(\beta, r)$ is not difficult because $\zeta = 0$ is not a branch point (as it might first appear) and the singularities of the integrand of (32) (treated as a function of the complex variable z) are simple poles located at

$$z = i\, \alpha_n^2/a^2, \quad n = 1, 2, \ldots$$

where α_n are the positive roots of

$$\tan \alpha = \alpha/ah \qquad (33)$$

Using an appropriate contour, we find

$$W(\beta, r) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{2\alpha_n \sin \alpha_n r/a \,.\, \exp[-\alpha_n^2 \beta]}{a(1 - ah)\cos \alpha_n - a\, \alpha_n \sin \alpha_n}, & \beta > 0 \\ 0, & \beta < 0 \end{cases}$$

Thus

$$u(r, \tau) = - 2\, ah \sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n r/a}{a(1 - ah)\cos \alpha_n - a\, \alpha_n \sin \alpha_n} \times \int_0^{\infty} \exp\left[- \alpha_n^2 \frac{\beta}{a^2}\right] T(\tau - \beta)\, d\beta \qquad (34)$$

Transforming back to $v(r, \tau)$ we have

$$v(r, \tau) = - 2\, h \sum_{n=1}^{\infty} \frac{\alpha_n \sin \alpha_n r/a}{r\, \{(1 - ah)\cos \alpha_n - \alpha_n \sin \alpha_n\}} \times \int_0^{\infty} \exp\left[- \alpha_n^2 \frac{\beta}{a^2}\right] T(\tau - \beta)\, d\beta \qquad (35)$$

It is easy to show that (35) is the solution to our problem by substituting in (27) and carrying out the indicated operations. We now require the mean square temperature in the sphere. It is obtained from (35) and is given by

$$\langle v^2(r, \tau)\rangle = 4\, h^2 \sum_{n, m=1}^{\infty} \frac{\alpha_n\, \alpha_m \sin \alpha_n (r/a) \sin \alpha_m (r/a)}{r^2 [(1 - ah)\cos \alpha_m - \alpha_n \sin \alpha_n]\, [(1 - ah)\cos \alpha_m - \alpha_m \sin \alpha_m]} \times$$
$$\int_0^{\infty}\int_0^{\infty} \exp[-(\alpha_n^2 \beta_1 + \alpha_m^2 \beta_2)/a^2]\, E\{T(\tau - \beta_1)\, T(\tau - \beta_2)\}\, d\beta_1\, d\beta_2 \qquad (36)$$

and for

(*i*) *Purely random ambient temperature*

$$E\{T(\tau - \beta_1)\, T(\tau - \beta_2)\} = S_o\, \delta(\beta_2 - \beta_1), \; S_o \text{ is a constant,}$$

we find

$$\langle v^2(r)\rangle = \frac{4\, S_o\, (aR)^2}{a^2} \left(\frac{a}{r}\right)^2 \sum_{n, m=1}^{\infty} \frac{\alpha_n\, \alpha_m}{\alpha_n^2 + \alpha_m^2}$$
$$\left\{\frac{\sin \alpha_n (r/a)}{(1 - ah)\cos \alpha_n - \alpha_n \sin \alpha_n} - \frac{\sin \alpha_m (r/a)}{(1 - ah)\cos \alpha_m - \alpha_m \sin \alpha_m}\right\} \qquad (37)$$

Equation (37) is plotted in Fig. 4 for several values of *ah*.

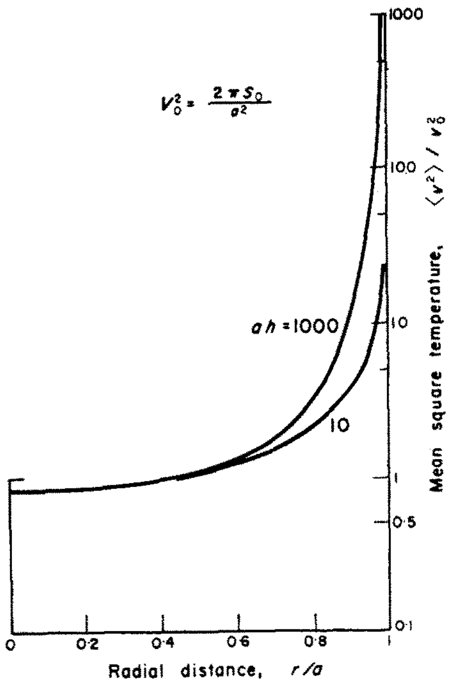

FIG. 4. Mean square temperature in a sphere with surface radiation into a medium with random temperature fluctuation.

(*ii*) *Markoff autocorrelation for the ambient temperature*

$$E\,\{T(\tau - \beta_1)\,T(\tau - \beta_2)\} = \langle T_o^2\rangle \exp[-\beta|\beta_2 - \beta_1|],\ \langle T_o^2\rangle \text{ and } \beta \text{ are constants,}$$

we find

$$\langle v^2(r)\rangle = 4\,a^2\,\langle T_o^2\rangle\,h^2\left(\frac{a}{r}\right)^2 \sum_{n,\,m=1}^{\infty} \frac{\alpha_n\,\alpha_m}{\alpha_n^2 + \alpha_m^2}\left\{\frac{1}{(\alpha_m^2 - \beta a^2)(\alpha_n^2 + \beta a^2)} - \frac{1}{(\alpha_n^2 + \alpha_m^2)(\alpha_m^2 - \beta a^2)}\right.$$
$$\left. + \frac{1}{(\alpha_n^2 + \alpha_m^2)(\alpha_m^2 + \beta a^2)}\right\}\cdot\frac{\sin\alpha_n(r/a)}{[(1 - ah)\cos\alpha_n - \alpha_n\sin\alpha_n]}\cdot\frac{\sin\alpha_m(r/a)}{[(1 - ah)\cos\alpha_n - \alpha_n\sin\alpha_n]} \tag{38}$$

(*b*) *Sphere with random surface temperature*

In this case the boundary value problem is

$$\left.\begin{aligned} \frac{\partial^2\{rv(r,\tau)\}}{\partial r^2} &= \frac{\partial\{rv(r,\tau)\}}{\partial\tau} \\ v(a,\tau) &= T(\tau) \\ v(r,\tau) &\text{ is bounded at } r = 0. \end{aligned}\right\} \tag{39}$$

Again substituting $u(r, \tau) = rv(r, \tau)$, we have

$$\left.\begin{aligned} \frac{\partial^2 u}{\partial r^2} &= \frac{\partial u}{\partial\tau} \\ u(a,\tau) &= aT(\tau) \\ u(0,\tau) &= 0 \end{aligned}\right\} \tag{40}$$

Taking Fourier transforms with respect to τ, we find

$$\bar{u}(r,\zeta) = \frac{a\bar{T}(\zeta)\sinh\sqrt{(i\zeta)}r}{\sinh\sqrt{(i\zeta)}a} \tag{41}$$

where $\bar{u}(r, \zeta)$ and $\bar{T}(\zeta)$ are defined just as in paragraph (*a*) above.

By the convolution theorem

$$u(r, \tau) = a \int_{-\infty}^{\infty} T(\tau - \lambda)\, W(\lambda, r)\, d\lambda \tag{42}$$

where

$$W(\lambda, r) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp[i\zeta\lambda] \frac{\sinh \sqrt{(i\zeta)}r}{\sinh \sqrt{(i\zeta)}a}\, d\zeta$$

$$= \begin{cases} -\dfrac{2}{a} \displaystyle\sum_{n=1}^{\infty} \left(\frac{n\pi}{a}\right) \exp\left[-\left(\frac{n\pi}{a}\right)^2 \lambda\right] \frac{\sin n\pi(r/a)}{\cos n\pi}, & \lambda > 0 \\ 0, & \lambda < 0 \end{cases} \tag{43}$$

Putting (43) in (42) we have

$$u(r, \tau) = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n\pi}{a}\right) \sin n\pi(r/a) \cdot \int_0^{\infty} \exp\left[-\left(\frac{n\pi}{a}\right)^2 \lambda\right] T(\tau - \lambda)\, d\lambda$$

Therefore

$$v(r, \tau) = \frac{2}{a}\left(\frac{a}{r}\right) \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n\pi}{a}\right) \sin n\pi(r/a) \cdot \int_0^{\infty} \exp\left[-\left(\frac{n\pi}{a}\right)^2 \lambda\right] T(\tau - \lambda)\, d\lambda \tag{44}$$

The mean square temperature is given by

$$\langle v^2(r, \tau)\rangle = \frac{4}{a^2}\left(\frac{a}{r}\right)^2 \sum_{n,m=1}^{\infty} (-1)^{n+m} \left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right) \sin\frac{n\pi r}{a} \sin\frac{m\pi r}{a} \times$$

$$\int_0^{\infty}\int_0^{\infty} \exp\left[-\left(\frac{n\pi}{a}\right)^2 \lambda_1 - \left(\frac{n\pi}{a}\right)^2 \lambda_2\right] E\{T(\tau - \lambda_1)\, T(\tau - \lambda_2)\}\, d\lambda_1\, d\lambda_2 \tag{45}$$

(*i*) *Purely random process for* $T(\tau)$

$$E\{T(\tau - \lambda_1)\, T(\tau - \lambda_2)\} = S_o \delta(\lambda_2 - \lambda_1), \quad S_o \text{ is a constant,}$$

we find

$$\langle v^2(r)\rangle = \frac{4S_o}{a^2}\left(\frac{a}{r}\right)^2 \sum_{n,m=1}^{\infty} (-1)^{n+m} \frac{nm}{n^2 + m^2} \sin\frac{n\pi r}{a} \sin\frac{m\pi r}{a}$$

$$= \frac{2\pi S_o}{a^2}\left(\frac{a}{n}\right)^2 \sum_{m=1}^{\infty} (-1)^{m+1}\, m\, \frac{\exp[m\pi r/a] - \exp[-m\pi r/a]}{\exp[m\pi] - \exp[-m\pi]} \sin\frac{m\pi r}{a} \tag{46}$$

Equation (46) is plotted in Fig. 5.

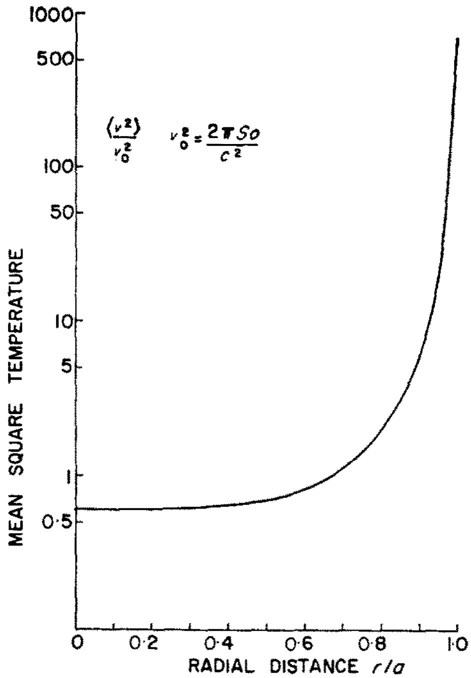

FIG. 5. Mean square temperature in a sphere with purely random surface temperature.

We see from the plot that the amplitude of the temperature fluctuations are sensibly confined to a thin surface layer. This is to be compared with the results found by Sommerfeld [18] in the study of the problem of the earth's temperature. He shows that rapid temperature fluctuations do not penetrate far into the earth. Fluctuations of very long periods may penetrate quite deeply.

(*ii*) *Markoff process for* $T(\tau)$

$$E\{T(\tau - \lambda_1)\,T(\tau - \lambda_2)\} = \langle T_o^2\rangle \exp\left[-\beta\,|\,\lambda_2 - \lambda_1\,|\right]$$

we get

$$\langle v^2(r)\rangle = \frac{\langle T_o^2\rangle}{a^2}\sum_{n,m=1}^{\infty}\left(\frac{a}{r}\right)^2\left(\frac{n\pi}{a}\right)\left(\frac{m\pi}{a}\right)\left\{\frac{1}{[(m\pi/a)^2 - \beta][(n\pi/a)^2 + \beta]}\right.$$

$$\left. + - \frac{1}{[(n\pi/a)^2 + (m\pi/a)^2][(m\pi/a)^2 - \beta]} + \frac{1}{[(n\pi/a)^2 + (m\pi/a)^2][(m\pi/a)^2 + \beta]}\right\} \times$$

$$\sin\frac{n\pi r}{a}\sin\frac{m\pi r}{a}. \qquad (47)$$

(*iii*) *Sphere with random generation of heat.* This problem is mathematically equivalent to the problem in a Section 3, paragraph (*a*).

5. TRANSIENT RANDOM HEAT CONDUCTION

To illustrate the methods of solving transient random heat conduction problems we solve a simple problem. Although stationarity cannot be claimed in such problems, we encounter no difficulty if we perform all our averages over the ensemble of functions that constitute the random process.

Consider the problem of an infinite slab as shown in Fig. 1 in which the face $x = l$ is maintained at the temperature zero while the temperature of the face $x = 0$ is suddenly increased to a temperature $F(t) = v(0, t)$ which is a random function. We would like to see how the mean square temperature in the slab evolves with time. Our problem is to solve

$$\left.\begin{aligned} &\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial \tau}, \qquad \tau = kt, \\ &v(0, \tau) = \begin{cases} F(\tau), & \tau > 0 \\ 0, & \tau < 0 \end{cases} \\ &v(l, \tau) = 0 \\ &v(x, 0) = 0 \end{aligned}\right\} \tag{48}$$

A formal solution to this problem (now using the Laplace transform as the appropriate analytical tool) is

$$v(x, \tau) = \int_0^\tau W(\tau - \lambda) F(\lambda)\, d\lambda \tag{49}$$

where

$$W(\tau) = \begin{cases} \dfrac{2\pi}{l^2} \displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} n \exp\left[-\left(\frac{n\pi}{l}\right)^2 \tau\right] \sin\frac{n\pi(l-x)}{l}, & \tau > 0 \\ 0, & \tau < 0 \end{cases} \tag{50}$$

We calculate the mean square temperature in the slab from (49) obtaining

$$\langle v^2(r, \tau)\rangle = \frac{4}{l^2} \sum_{n,m=1}^{\infty} (-1)^{n+m} \left(\frac{n\pi}{l}\right)\left(\frac{m\pi}{l}\right) \left\{\int_0^\tau \int_0^\tau \exp\left[-\left(\frac{n\pi}{l}\right)^2 - \left(\frac{m\pi}{l}\right)^2\right]\tau \times \exp\left[\left(\frac{n\pi}{l}\right)^2 \lambda_1 + \left(\frac{m\pi}{l}\right)^2 \lambda_2\right] R_F(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2\right\} \times \sin\frac{n\pi(l-x)}{l} \sin\frac{m\pi(l-x)}{l},$$

$$R_F(\lambda_1, \lambda_2) = E\{F(\lambda_1) F(\lambda_2)\} \tag{51}$$

If $F(\tau)$ is a purely random stationary process, that is,

$$R\ (\lambda_1, \lambda_2) = S_o \delta(\lambda_2 - \lambda_1), \quad S_o \text{ constant},$$

then (51) becomes

$$\langle v^2(r, \tau)\rangle = \frac{4S_o}{l^2} \sum_{n,m=1}^{\infty} (-1)^{n+m} \left\{\frac{(n\pi/l)(m\pi/l)}{(n\pi/l)^2 + (m\pi/l)^2} \cdot \{1 - \exp -[(n\pi/l)^2 + (m\pi/l)^2]\tau\}\right\} \times \sin\frac{n\pi(l-x)}{l} \sin\frac{m\pi(l-x)}{l} \tag{52}$$

We note that as $\tau \to \infty$, $\langle v^2(r, \tau)\rangle$ approaches the stationary value.

6. SPECTRAL ANALYSIS

Frequently, it is desirable to look at the average power distribution with frequency in both the input and output of linear systems. The linear system is in effect a filter that selects power from a certain band of frequencies in the input. Central to the spectral analysis of stationary random functions is the *average power spectral density* defined mathematically as the Fourier transform of the autocorrelation function. For example, if $x(t)$ is a stationary random function (having mean zero) with autocorrelation function $R_x(h) = E\{x(t)\,x(t+h)\}$, the spectral density of $x(t)$ would be given by

$$S_x(\omega) = \int_{-\infty}^{\infty} \exp\left[-i\omega h\right] R_x(h)\,\mathrm{d}h \qquad (53)$$

ω = angular frequency.

It is easy to show [13] that the average power spectral density gives the power density distribution with frequency. The power spectral density of the temperature in the problem of Section 3, paragraph (*a*)(*i*) above (R_v given by equation (11) with $\xi = x$) is found to be

$$S_v(\omega) = \frac{S_0}{l\kappa^2\rho^2c^2}\sum_{n=1}^{\infty}\frac{1}{\omega^2 + \kappa^2(n\pi/l)^4}\sin^2\frac{n\pi x}{l} \qquad (54)$$

Fig. 6 shows a plot of the first term approximation to $S_v(\omega)$ at $x/l = 1/2$.

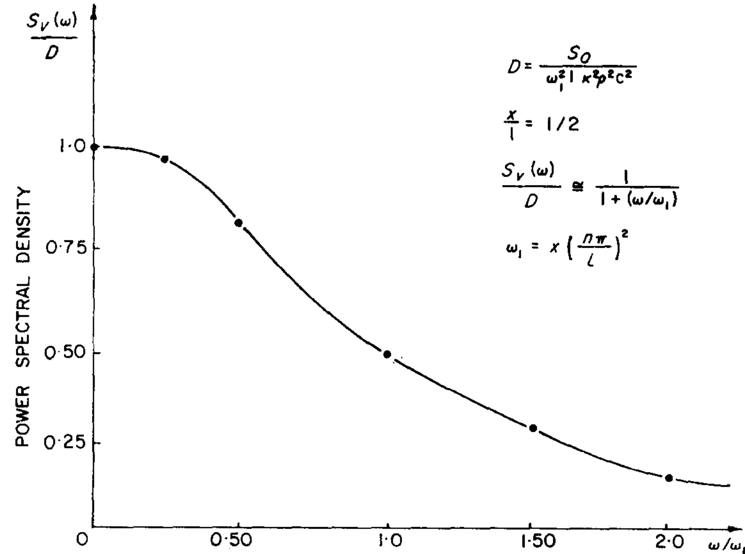

FIG. 6. Spectral density at the midpoint of the slab.

7. CONCLUDING REMARKS

This paper has shown that the methods of applied stochastics as employed in electrical communication technology and random vibration theory are readily adapted to problems of random heat conduction in solids. The time derivative term in the heat equation plays the same role as velocity damping in random vibration theory [6]. Convergence problems for the mean square temperature fields are not as severe as those for the mean square stress fields in random vibration problems [8]. We may, however, expect some convergence problems for the mean square heat flux for extremely wide-band random temperature disturbances. This may arise from the fact that we must differentiate the temperature series with respect to the space variables to get the heat flux.

Finally, we remark that our solution methods include as a special case the problem treated by J. Dreyfus [19] concerning heat transmission through building walls with periodic variation of external temperatures.

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Résumé—Plusieurs problèmes théoriques de conduction de la chaleur dans les solides ont été étudiés, problèmes dans lesquels la chaleur engendrée à l'intérieur et/ou la température extérieure sont des fonctions stochastiques de la position et du temps. Pour des systèmes linéaires sujets à des peturbations aléatories gaussiennes, il est suffisant d'avoir la valeur moyenne et la moyenne quadratique du champ de température pour déterminer complètement l'étude statistique.

Cette façon de procéder a été poursuivie dans les quelques problèmes examinés. On trouve que les problèmes de convergence pour les séries infinies représentant la moyenne quadratique de la température (avec une perturbation en bruit blanc) sont beaucoup moins sévères que ceux rencontrés dans la thèorie des vibrations aléatoires où les équations aux dérivées partielles sont de type hyperbolique tandis qu'elles sont paraboliques dans la conduction de la chaleur. Des calculs numériques sont effectués dans quelques cas avec un calculateur numérique.

Quelques commentaires sur le problème de la température dans le sol ont été donnés.

Zusammenfassung—Es werden verschiedene Probleme der Wärmeleitung in festen Körpern untersucht, wobei innere Wärmequellen und/oder die Umgebungstemperatur stochastische Funktionene des Ortes und der Zeit sind. Bei linaren Systemen die willkürlichen Verteilungen nach einem Gausschen Prozess unterworfen sind genügt es den mittleren und den mittleren quadratischen Wert des Temperaturfeldes zu bestimmen um der vollständigen Statistik zu genügen. Dies ist für die verschiedenen behandelten Probleme durchgefürht. Es zeigt sich, dass die Konvergenzprobleme für die unendlichen Reihen der mittleren quadratischen Temperatur (unterhalb der weissen Lärmstörung) weit weniger schwer wiegen als die der Vibrationstheorie, deren bestimmende partielle Differentialgleichungen hyperbolischen Charakter besitzen im Gegensatz zu den parabolischen der Wärmeleitung. Numerische Berechnungen wurden für einige Fälle mit einer Digitalrechenmaschine durchgefürht.

Hinweise auf das Problem der Erdtemperatur sind gegeben.

Аннотация—Рассмотрены несколько задау теплопроводности твердых тел, когда внутреннее тепловыделение и/или температура окружающей среды являются стохастическими функциями координат и времени. Для линейных систем, подверженных произвольным возмущениям, приводящим к процессу, описываемому законом Гаусса, достаточно определить среднюю и средне-квадратичную величину температурного поля, чтобы установить его полную статистику. Эта операция выполнялась в нескольких из рассмотренных задач. Было показано, что задачи сходимости бесконечных рядов средне-квадратичной температуры (при бело-шумовых возмущениях) гораздо менее строги, чем задачи, встречающиеся в теории протзвольной вибрации, где основные уравнения в частных производных являются уравнениями гиперболического типа, а не параболического, как это имеет место в задачах теплопроводности. В некоторых случаях численные расчеты производились с помощью вычислительной машины.

Решение указанных задач позволило сделать некоторые замечания, относящиеся к задаче о температуре земли.